Alternative Solution to Basel Problem

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As we all know the $S = \sum_{1}^{\infty} \frac{1}{u^2}$ is equal to $S = \sum_{1}^{\infty} \frac{1}{u^2} = \int_0^{\infty} \frac{xdx}{e^x - 1}$. So in order to calculate the sum it is enough to calculate the integral. There is a relation between integral $I_1 = \int_0^{\infty} \frac{xdx}{e^x - 1}$ and $I_2 = \int_0^{\infty} \frac{xdx}{e^x + 1}$ such that $I_2 = \frac{1}{2} I_1$ which is easy to verify.

So let's transform the first integral by substitution $x = -ln\alpha$. I_1 is then equal to $I_1 = \int_0^1 \frac{ln\alpha}{\alpha - 1} d\alpha$. Similarly $I_2 = \int_1^0 \frac{ln\alpha}{\alpha + 1} d\alpha$. The sum $I_1 + I_2 = \frac{3I_1}{2} = 2\int_0^1 \frac{ln\alpha d\alpha}{\alpha^2 - 1}$ so instead of integral I_1 let's calculate the right hand-side.

We introduce a function:

$$I(x) = \int_0^1 \frac{ln(1 + (\alpha^2 - 1)sin^2x)}{\alpha^2 - 1} d\alpha \tag{1}$$

$$I'(x) = \int_0^1 \frac{(\alpha^2 - 1) \cdot 2cosx \cdot sinx}{(\alpha^2 - 1)} \cdot \frac{1}{(\alpha^2 - 1)sin^2x + 1} d\alpha \tag{2}$$

$$I'(x) = \frac{sin2x}{sin^2x} \int_0^1 \frac{d\alpha}{\alpha^2 + \frac{1}{sin^2x - 1} - 1} \tag{3}$$

$$I'(x) = \frac{2cosx}{sinx} \int_0^1 \frac{d\alpha}{\alpha^2 + \frac{cos^2x}{sin^2x}}, \qquad a = \frac{1}{\tan x} \tag{4}$$

$$I'(x) = \frac{2cosx}{sinx} \frac{1}{a} \arctan \frac{1}{a} \alpha \Big|_0^1 \tag{5}$$

$$I'(x) = \frac{2cosx}{sinx} \cdot \frac{sinx}{cosx} \cdot \arctan(\tan x) \tag{6}$$

$$I'(x) = 2x, \qquad I(x) = x^2 + c \tag{7}$$

$$I(0) = 0, \qquad c = 0 \tag{8}$$

$$I\left(\frac{\pi}{2}\right) = 2\int_0^1 \frac{ln\alpha d\alpha}{\alpha^2 - 1} = \frac{\pi^2}{4} \tag{9}$$

$$\int_0^1 \frac{ln\alpha}{\alpha^2 - 1} = \frac{\pi^2}{8} \tag{10}$$

$$\frac{1}{\alpha^2 - 1} = \frac{1}{2}\left(\frac{1}{\alpha - 1} - \frac{1}{\alpha + 1}\right) \tag{11}$$

$$\int_0^1 \frac{ln\alpha d\alpha}{\alpha^2 - 1} = \int_0^1 \frac{1}{2}\left(\frac{ln\alpha}{\alpha - 1} - \frac{ln\alpha}{\alpha + 1}\right) d\alpha \tag{12}$$

$$\frac{1}{2} \cdot \frac{3}{\cancel{2}} I_1 = \frac{\pi^2}{\cancel{8}} \tag{13}$$

$$I_1 = \frac{\pi^2}{6} \tag{14}$$